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NECKING OF AN ANISOTROPIC PLASTIC MATERIAL

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Abstract—The two conditions for stable necking, treated as a bifurcation of stress path, are derived. The anisotropic material considered is one for which the plastic work increment = $\bar{\sigma}d\bar{e}$, where both the generalized yield stress $\bar{\sigma}$ and generalized plastic strain increment $d\bar{e}$ are invariant functions. The method is applied to the necking of a thin cylinder under internal pressure.

The problem of necking of a ductile strain-hardening material may be approached in two ways: In the first,[1], the possibility of a bifurcation of the strain and stress paths is considered; in the second,[2], we consider the possibility of a bifurcation of the stress path. The former requires consideration of a kinematically admissible velocity field in a post-necking mode of deformation; the method is not simple to apply and few exact solutions are available. The latter method requires only the consideration of a statically admissible stress increment field for material instantaneously in the pre-necking mode. The method may be more restricted in application, but is simpler. It will be one purpose of the present paper to attempt a more fundamental justification of the stress-path bifurcation technique than is presently available.

A previous attempt to consider the necking of anisotropic material[3] has assumed the validity of Hill's theoretical description of strain hardening[4]. However recent work[5] suggests that the definition of generalised plastic strain $\bar{e}$ requires modification, but that it may be replaced by one which is both invariant with respect to rotation of the coordinate axes of reference and satisfies the work condition.

$$\sigma_{ij}\,de_{ij} = \bar{\sigma}d\bar{e} \geq 0 \tag{1}$$

where σ_{ij} is the stress field tensor, de_{ij} the associated plastic strain increment tensor field and $\bar{\sigma}$ the generalised yield stress. In what follows elastic strains will be assumed to be negligible.

Consider the deformation of a volume V, enclosed by the surface A. Let F_i be the components of surface tractions on A and σ_{ij} the stress tensor field satisfying the equations of equilibrium in V and the stress boundary conditions on A. Let de_{ij} be the kinematically admissible plastic strain increment tensor field associated with the σ_{ij} through the constitutive equation for the material. We suppose that, in the interval considered, the surface A is subjected to surface tractions dF_i prescribed by the loading conditions on a part A_F of A, and by displacement conditions on the remaining part A_u. In addition there may exist inadvertent disturbing forces dF^D on A_F. Let $d\sigma_{ij}$ be the stress increment field in equilibrium with $(dF_i + dF_i^D)$ on A_F, satisfying stress equilibrium in V, and associated with the de_{ij}

through the current rate of strain hardening. Then, for an initially quasi-static deformation the equation of virtual work may be written

$$dW^D = (dE - dW) + dK \tag{2}$$

where dW^D, and dW are the work increments done by the dF_i^D and the dF_i respectively, $dK \gtrless 0$ the increase in kinetic energy and

$$dE = \frac{1}{2}\int_V d\sigma_{ij}\, de_{ij}\, dV. \tag{3}$$

From equation (2) it may be seen that all the work dW^D will be converted to kinetic energy if $(dE - dW) \leq 0$, no matter how small the dW^D. Conversely, if $(dE - dW) > 0$, the work done by disturbing forces will be converted to kinetic energy only if the dW^D is sufficiently large. A necessary condition for stability is therefore

$$(dE - dW) > 0. \tag{4a}$$

Further, if $dE \leq 0$, all the work done by prescribed external forces dF_i will be converted to kinetic energy and the deformation is unstable. That is, for stable quasi-static deformation we must have

$$dE > 0. \tag{4b}$$

The smallest possible rate of energy of dissipation dE satisfying both inequalities (4) corresponds to $dW \leq 0$. From equation (1), since both dE and dK are non-negative then, if dW^D is negligibly small, dW is non-negative. Hence the least possible admissible rate of energy dissipation satisfying both inequalities (4) corresponds to

$$dW \equiv \int_A (dF_i\, du_i) dA = 0 \tag{5}$$

where du_i is the actual displacement field vector.

We consider now the possibility of stress increments $d\sigma_{ij}^0$ which satisfy all given conditions on the $d\sigma_{ij}$, excepting that the former is not necessarily consistent with the current rate of strain hardening. Virtual work for the $d\sigma_{ij}^0$ in the strain field de_{ij} is

$$dW^D = (dE^0 - dW) + dK^0$$

where

$$dE^0 \equiv \frac{1}{2}\int_V (d\sigma_{ij}^0 de_{ij}) dV$$

and $dK^0 > 0$ is the corresponding kinetic energy. In the absence of significant disturbing forces the transition $d\sigma_{ij}^0 \rightarrow d\sigma_{ij}$ in stress space is stable, and the reverse transition unstable (or neutral) if $(dK - dK^0) \geq 0$. That is if $(dE^0 - dE) \geq 0$, or

$$\int_V (d\sigma_{ij}^0 - d\sigma_{ij}) de_{ij}\, dV \geq 0. \tag{6}$$

Equation (5) and inequality (6) are the necessary and sufficient conditions for a stable bifurcation of the stress path. At the instant of bifurcation the equality in equation (5) is just satisfied; this will be taken to be one criterion for the onset of necking.

The de_{ij} will be derived from a positive, homogeneous, invariant, potential function $F(\sigma_{ij})$ forming a convex surface enclosing the origin in stress space. Thus

$$de_{ij} = d\lambda \frac{\partial F}{\partial \sigma_{ij}} \tag{7}$$

where $d\lambda$ is a positive scalar factor, and

$$\sigma_{ij} \frac{\partial F}{\sigma \partial_{ij}} \equiv F \tag{8}$$

$$\sigma_{ij} de_{ij} \equiv F d\lambda \geq 0$$

where[5]

$$d\lambda = +(de_{ij} de_{ij})^{1/2} [(\partial F/\partial \sigma_{Kl})(\partial F/\partial \sigma_{Kl})]^{1/2}. \tag{9}$$

The resultant plastic strain increment is then directed along the outward normal to the potential surface. The yield function $\bar{\sigma}$ will be taken to coincide with the plastic potential and $d\lambda$ will be taken to be a suitable definition of generalised plastic strain increment $d\bar{e}$. Thus

$$\bar{\sigma} = F, \qquad \text{and} \qquad d\bar{e} = d\lambda \tag{10}$$

and the work condition of equation (1) is automatically satisfied.

We consider the possibility of necking at a cross-section across which the stresses are uniformly distributed. Then, using equation[1] the necking criterion (6) may be written[2]

$$\frac{1}{Z} \equiv \frac{1}{\bar{\sigma}} \frac{d\bar{\sigma}^0}{d\bar{e}} \geqslant \frac{1}{\bar{\sigma}} \frac{d\bar{\sigma}}{d\bar{e}}. \tag{11}$$

Here $d\bar{\sigma}/d\bar{e}$ is the current rate of strain-hardening and Z is the critical value of the subtangent to the strain-hardening curve. The latter may be evaluated as follows:
Consider the stresses $\sigma_{\alpha\beta}$ on areas A_α to be related to loads $T_{\alpha\beta}$ by

$$T_{\alpha\beta} = A_\alpha \sigma_{\alpha\beta}. \tag{12}$$

(no sum over α, β)
Then, making use of the following identities from equations (1), (7) and (10)

$$\frac{d\bar{\sigma}^0}{d\bar{e}} = \frac{\partial F}{\partial \sigma_{kl}} \frac{d\sigma_{kl}^0}{d\bar{e}}$$

and

$$\frac{d\sigma_{kl}^0}{d\bar{e}} = \frac{d\sigma_{\alpha\beta}^0}{d\bar{e}} \delta_{ak} \delta_{\beta l}$$

where $\delta_{\alpha k}$ is the Kronecker delta, together with equation (7) and differentiation of equation (12), we find

$$Z^{-1} = \frac{1}{\bar{\sigma}} \left[\frac{1}{A_\alpha} \frac{\partial T_{\alpha\beta}}{\partial e_{ij}} - \frac{\sigma_{\alpha\beta}}{A_\alpha} \frac{\partial A_\alpha}{\partial e_{ij}} \right] \left[\frac{\partial F}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{kl}} \delta_{k\alpha} \delta_{l\beta} \right]. \tag{13}$$

We consider here the von Mises[6] plastic potential $F^2 = \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl}$, together with Hill's assumptions for the orthotropic plastic material[4]. The conditions that must then be satisfied by the anisotropic coefficients are discussed elsewhere[2, 4, 5] but may be restated briefly as follows:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

$$C_{1123} = C_{1221} = \cdots = 0$$

and

$$C_{11222} = -\tfrac{1}{2}(C_{1111} + C_{2222} - C_{3333}),$$

For simplicity, we shall set

$$F \equiv -C_{1122}, (G+H) \equiv C_{1111}, N = C_{1212}, \ldots$$

where F, G, H, L, M, N are proportional to the corresponding quantities defined by Hill. In particular, for plastic isotropy we would have $F = 1$, $N = 3$, and so on.

Thus, use of equations (7)–(10) leads to the folowing results:

Flow rule:

$$\begin{aligned} 2(\bar{\sigma}/d\bar{e})de_{11} &= H(\sigma_{11} - \sigma_{22}) + G(\sigma_{11} - \sigma_{33}) \\ &= \phi_1, \text{ say} \\ 2(\bar{\sigma}/d\bar{e})de_{12} &= N\sigma_{12}. \end{aligned} \tag{14}$$

Yield criterion:

$$\bar{\sigma} = +\{\tfrac{1}{2}[(G+H)(\sigma_{11} - \sigma_{22})(\sigma_{11} - \sigma_{33}) + \cdots + 2N\sigma_{12}^2 + \cdots]\}^{1/2}. \tag{15}$$

Generalised plastic strain increment:

$$d\bar{e} = +(\psi)^{-1}(2/3 de_{ij} de_{ij})^{1/2} \tag{16}$$

where

$$\psi^2 = +1/6\{[(G+H)^2 + G^2 + H^2](\sigma_{11}/\bar{\sigma})^2 + \cdots - 2[H(F+G+2H) - FG] \\ (\sigma_{11}/\bar{\sigma})(\sigma_{22}/\bar{\sigma}) + 2N^2(\sigma_{12}/\bar{\sigma})^2 + \cdots\}. \tag{17}$$

Necking condition:

$$Z^{-1} = (4\bar{\sigma}^3)^{-1}\left\{\left(\frac{1}{A_1}\frac{\partial T_{11}}{de_{11}} - \frac{\sigma_{11}}{A_1}\frac{\partial A_1}{de_{11}}\right)\phi_1{}^2 + \left(\frac{1}{A_1}\frac{\partial T_{11}}{\partial e_{22}} - \frac{\sigma_{11}}{A_1}\frac{\partial A_1}{\partial e_{22}}\right)\phi_2{}^2 + \cdots \right. \\ \left. + \left(\frac{1}{A_2}\frac{\partial T_{22}}{\partial e_{11}} - \frac{\sigma_{22}}{A_2}\frac{\partial A_2}{\partial e_{11}}\right)\phi_1{}^2 + \cdots + \left(\frac{1}{A_1}\frac{\partial T_{12}}{\partial e_{12}} - \frac{\sigma_{12}}{A_1}\frac{\partial A_2}{\partial e_{12}}\right)N^2\sigma_{12}{}^2 + \cdots\right\}. \tag{18}$$

We have thus justified our previous assumption[2] that the principle of maximum plastic work is just violated at the onset of necking. The additional criterion of equation (5) allows us to establish whether certain loads or pressures will be a maximum of necking; this replaces previous arbitrary assumptions. Finally, the use of an invariant form of the generalised plastic strain increment should allow better agreement with experiment.

EXAMPLE

We consider the necking failure of a long thin tube of length l, radius $r \ll l$, wall thickness $t \ll r$, subjected to a uniform internal pressure p. The corresponding strain increments are:

$$de_{11} = \frac{dr}{r}; \; de_{22} = \frac{dl}{l}; \; de_{33} = \frac{dr}{t}.$$

The first necking condition, equation (5) gives

$$dW = (\pi r^2 dl + 2\pi r l dr)\ dp = 0$$
$$\text{or} \quad dp = 0$$

and the pressure is a maximum at the onset of necking.

The second necking condition determined by equations (11) and (18) may be evaluated as follows:

Circumferential and axial stresses are defined by the loads and areas

$$T_{11} = 2rlp \qquad A_1 = 2lt$$
$$T_{22} = \pi r^2 p \qquad A_2 = 2\pi rt$$
$$T_{33} = 0.$$

Thus

$$\sigma_{22} = \frac{1}{2}\sigma_{11} = \frac{pr}{2t}, \sigma_{33} = 0.$$

The corresponding non-zero rates of change of loads and areas are

$$\frac{1}{A_1}\frac{\partial T_{11}}{\partial e_{11}} = \frac{1}{A_1}\frac{\partial T_{11}}{\partial e_{22}} = \frac{1}{A_2}\frac{\partial T_{22}}{\partial e_{22}} = \sigma_{11}$$
$$\frac{1}{A_1}\frac{\partial A_1}{\partial e_{11}} = \frac{1}{A_2}\frac{\partial A_2}{\partial e_{22}} = -1.$$

The yield criterion, equation (15) becomes

$$(\sigma_{11}/\bar{\sigma}) = (8)^{1/2}(4G + H + F)^{-1/2}$$

and the corresponding flow rule, equation (14) gives

$$\phi_1 = \tfrac{1}{2}\sigma_{11}(2G + H), \qquad \phi_2 = \tfrac{1}{2}\sigma_{11}(F - H).$$

The corresponding value of the critical subtangent z to the generalised strain-hardening curve is

$$z = (2)^{1/2}(4G + H + F)^{3/2}[4(2G + H)^2 + 5(H - F)^2]^{-1}.$$

In particular, for isotropic material, $F = G = H = 1$, and $z = (3)^{-1/2}$.

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